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On the Wronskian of the hypergeometric functions
of type $(n+1, m+1)$

by Michitake Kita

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Introduction

Many specialists believe that the rank of the hypergeometric system $E(n+1, m+1)$ is equal to $\binom{m-1}{n}$. Although this fact is fundamental in studying the hypergeometric functions, there is no rigorous proof of the fact in general case: $\lambda_j \notin \mathbb{Z}$ ($1 \leq j \leq m$), $\sum \lambda_j \notin \mathbb{Z}$. This is the brief resume of [K] in which we gave a proof of the fact based on the perfect pairing of certain twisted rational de Rham cohomology and twisted homology associated with the hypergeometric integral of type $(n+1, m+1)$.

§1. The hypergeometric function (HGF) of type $(n+1, m+1)$ ($n < m$)

Let

$$W_{n+1, m+1} := \left\{ w = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0m} \\ \vdots & & & \\ w_{n0} & w_{n1} & \cdots & w_{nm} \end{pmatrix} \in M(n+1, m+1, \mathbb{C}) \mid \begin{array}{l} \text{rank } w = n+1 \\ \text{each} \\ \text{column} \neq 0 \end{array} \right\},$$

$$[t_0 : t_1 : \cdots : t_n] \in \mathbb{CP}^n,$$

$$\tau := \sum_{i=0}^n (-1)^i t_i dt_0 \wedge \cdots \wedge \hat{dt}_i \wedge \cdots \wedge dt_n : \text{ an } n\text{-form on } \mathbb{C}^{n+1}.$$

Let $\tilde{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_m) \in (\mathbb{C} \setminus \mathbb{Z})^{m+1}$ be exponents with the condition

$$(1) \quad \sum_{j=0}^m \lambda_j + n + 1 = 0.$$

Then, for each $w \in W_{n+1,m+1}$, the n -form

$$\prod_{j=0}^m (w_{0j}t_0 + \dots + w_{nj}t_n)^{\lambda_j \cdot \tau}$$

can be seen as an n -form on \mathbb{CP}^n because of (1). Taking a suitable twisted cycle σ as a domain of integration, we define a function by the integral

$$\Phi(\tilde{\lambda}, w) = \int_{\sigma} \prod_{j=0}^m \left(\sum_{i=0}^n w_{ij} t_i \right)^{\lambda_j \cdot \tau},$$

which will be called the *HG integral* (or function) of type $(n+1, m+1)$

$$\text{The groups } GL(n+1, \mathbb{C}) \text{ and } H_{m+1} := \left\{ \begin{pmatrix} h_0 & & 0 \\ & h_1 & \\ & & \ddots \\ 0 & & & h_m \end{pmatrix} \right\} \text{ act on}$$

$W_{n+1,m+1}$ from left and right, respectively as $w \longrightarrow gwh$. $\Phi(\tilde{\lambda}, w)$ is homogeneous under the above two kind of group actions:

$$(2) \begin{cases} \Phi(\tilde{\lambda}, gw) = \frac{1}{\det g} \Phi(\tilde{\lambda}, w) & \text{for } g \in GL(n+1) \\ \Phi(\tilde{\lambda}, wh) = \prod_{j=0}^m h_j^{\lambda_j} \Phi(\tilde{\lambda}, w) & \text{for } h \in H_{m+1}. \end{cases}$$

Since

$$\frac{\partial^2 \Phi}{\partial w_{ip} \partial w_{jq}} = \lambda_p \lambda_q \int_{\sigma} \frac{t_p t_q}{\left(\sum_i w_{ip} t_i \right) \left(\sum_j w_{jq} t_j \right)} \prod_{j=0}^m \left(\sum_i w_{ij} t_i \right)^{\lambda_i \cdot \tau},$$

we have

$$\frac{\partial^2 \Phi}{\partial w_{ip} \partial w_{jq}} = \frac{\partial^2 \Phi}{\partial w_{iq} \partial w_{jp}} \quad (0 \leq i, j \leq n, 0 \leq p, q \leq m).$$

Hence, passing to the Lie algebra version of (2), we have that

$\Phi(\tilde{\lambda}, w)$, viewed as a function on $W_{n+1, m+1}$, satisfies the following system $E(n+1, m+1; \tilde{\lambda})$ of differential operators:

$$(*) \quad \begin{cases} \sum_{i=0}^n w_{ip} \frac{\partial \Phi}{\partial w_{ip}} = \lambda_p \Phi & (0 \leq p \leq m) \quad (H - \text{homogeneity}) \\ \sum_{p=0}^m w_{ip} \frac{\partial \Phi}{\partial w_{jp}} = -\delta_{ij} \Phi & (1 \leq i, j \leq n) \quad (GL(n+1)\text{-homogeneity}) \\ \begin{vmatrix} \partial_{ip} & \partial_{iq} \\ \partial_{jp} & \partial_{jq} \end{vmatrix} \Phi = 0 & (0 \leq i, j \leq n, 0 \leq p, q \leq m) \end{cases}$$

We set

$$W'_{n+1, m+1} = \left\{ w \in W_{n+1, m+1} \mid \begin{array}{l} \text{those matrices of which every minor of} \\ \text{order } n+1 \text{ in } w \text{ is non-zero} \end{array} \right\}$$

which is an open dense subset of $W_{n+1, m+1}$. For each $w \in W'_{n+1, m+1}$

there exist $g \in GL(n+1)$ and $h \in H_{m+1}$ such that

$$g \cdot w \cdot h = \begin{pmatrix} 1 & & & 1 & & & & 1 & & & 1 \\ & 1 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & z_{1, n+1} & & & & & & \\ & & & & \vdots & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & z_{n, n+1} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & z_{n, m-1} & & \\ & & & & & & & & & -1 & \end{pmatrix}.$$

Using $u_i = t_i/t_0$ ($1 \leq i \leq n$) the non-homogeneous coordinates

in CP^n , we set

$$\begin{cases} f_i(u) = u_i & (1 \leq i \leq n) \\ f_j(u) = 1 + z_{1j}u_1 + \cdots + z_{nj}u_n & (n+1 \leq j \leq m-1) \\ f_m(u) = 1 - \sum_{i=1}^n u_i. \end{cases}$$

From now on we keep this notations. Then the integral $\Phi(\tilde{\lambda}, w)$ takes

the following form

$$\bar{\Psi}(\lambda, z) = \int_{\sigma} \prod_{j=0}^m f_j(u)^{\lambda_j} \cdot du_1 \wedge \cdots \wedge du_n$$

where, by our assumptions, $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfies the conditions

$$\begin{cases} \lambda_j \notin \mathbb{Z} & (1 \leq j \leq m) \\ \sum_{j=1}^m \lambda_j \notin \mathbb{Z}. \end{cases}$$

If we take the integration over the twisted cycle $\Delta^n(\omega)$ associated with the n -simplex

$$\Delta^n = \{ u \in \mathbb{R}^n \mid 0 \leq u_i \ (1 \leq i \leq n), \sum u_i \leq 1 \}$$

$$\text{where } \omega = d \log \left\{ \prod_{i=1}^n u_i^{\lambda_i} (1 - \sum u_i)^{\lambda_m} \right\},$$

then we obtain the following power series expansion of $\bar{\Psi}(\lambda, z)$:

$$\bar{\Psi}(\lambda, z) = \text{const} \sum_{\nu} \frac{\prod_{i=1}^n (\alpha_i; \sum_j \nu_{ij}) \prod_j (\beta_j; \sum_i \nu_{ij})}{(\gamma; \sum_{i,j} \nu_{ij}) \nu!} z^{\nu}$$

where

$$\nu = \begin{pmatrix} \nu_{1,n+1} & \cdots & \nu_{1,m-1} \\ \vdots & & \\ \nu_{n,n+1} & \cdots & \nu_{n,m-1} \end{pmatrix} \in M(n, m-n-1; \mathbb{Z}_{\geq 0})$$

and

$$\alpha_i = \lambda_i + 1, \quad \beta_j = -\lambda_j \quad \& \quad \gamma = -\sum_{i=1}^n \lambda_i - \lambda_m - n.$$

§2. Twisted de Rham theory of HG integrals

Since, in case of $z_{ij} \in \mathbb{R}$, twisted cycles become *visible*, we assume, from now on, that

$$(H.2) \quad z_{ij} \in \mathbb{R} \quad \text{for } 1 \leq i \leq n, n+1 \leq j \leq m-1.$$

Set

$$H_j = \{u \in \mathbb{C}^n \mid f_j(u) = 0\}, \quad X = \mathbb{C}^n \setminus \bigcup_{j=1}^m H_j, \quad (H_j)_R = H_j \cap \mathbb{R}$$

and suppose that

$$(H.3) \quad m \text{ hyperplanes and the hyperplane at infinity are in general position in } \mathbb{CP}^n.$$

For simplicity of writing we set

$$U(u) = \prod_{j=1}^m f_j(u)^{\lambda_j},$$

$$\varphi\langle J \rangle = \varphi\langle j_1, \dots, j_p \rangle := \frac{df_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{df_{j_p}}{f_{j_p}}.$$

We know that

$$\#\{\text{bounded components of } \mathbb{R}^n \setminus \bigcup_j H_{jR}\} = \binom{m-1}{n}.$$

Taking a bounded chamber Δ with the standard orientation of \mathbb{R}^n together with a branch of $U(u)$, the following integral

$$\int_{\Delta} U(u) \varphi\langle j_1, \dots, j_n \rangle = \int_{\Delta} \prod f_j(u)^{\lambda_j} \cdot \frac{df_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{df_{j_n}}{f_{j_n}}$$

is called a *hypergeometric integral* where $\operatorname{Re} \lambda_j > 0$. Classical

examples suggest that $\binom{m-1}{n}$ hypergeometric integrals

$$\int_{\Delta} U(u) \varphi\langle 1, \dots, n \rangle \quad \text{taken over } \binom{m-1}{n} \text{ bounded chambers may be all}$$

linearly independent solutions of the HG system

$E(n+1, m+1; \lambda_0, \lambda_1-1, \dots, \lambda_n-1, \lambda_{n+1}+1, \dots, \lambda_m)$. To see this, it is necessary to study the twisted de Rham theory associated with the many-valued function $U(u)$. For $\varphi \in \Gamma(X, \mathcal{E}_X^p)$, we have $d(U\varphi) = U(d\varphi + \omega \wedge \varphi)$ on the universal covering manifold \tilde{X} where we set

$$\omega = \frac{dU}{U} = \sum_{j=1}^m \lambda_j \frac{df_j}{f_j}.$$

Hence instead of considering the exterior differentiation d on \tilde{X} , we are led to study the covariant differentiation

$$\nabla_\omega \varphi = d\varphi + \omega \wedge \varphi.$$

Let $\mathcal{G}_\omega = \{h \in \mathcal{O}_X \mid \nabla_\omega h = 0\}$ which is a complex local system of rank one. It is known the following results:

1. The comparison theorem.

$$H^p(X, \mathcal{G}_\omega) \simeq H^p(X, \nabla_\omega) := \frac{\left\{ \begin{array}{l} \nabla_\omega\text{-closed rational } p\text{-forms} \\ \text{which are holo. on } X \end{array} \right\}}{\nabla_\omega \left\{ \begin{array}{l} \text{rational } (p-1)\text{-forms} \\ \text{which are holo. on } X \end{array} \right\}}$$

2. $H^p(X, \nabla_\omega) = 0$ for $p \neq n$,

$$\begin{aligned} H^n(X, \nabla_\omega) &\simeq \frac{\{ \{ \varphi \langle j_1, \dots, j_n \rangle \mid 1 \leq j_1 < \dots < j_n \leq m \} \}}{\omega \wedge \{ \{ \varphi \langle j_1, \dots, j_{n-1} \rangle \mid 1 \leq j_1 < \dots < j_{n-1} \leq m \} \}}, \\ &\simeq \{ \{ \varphi \langle j_1, \dots, j_n \rangle \mid 1 \leq j_1 < \dots < j_n \leq m-1 \} \} \end{aligned}$$

and

$$\dim H^n(X, \nabla_\omega) = \binom{m-1}{n}.$$

3. Let \mathcal{G}_ω^\vee be the dual complex local system to \mathcal{G}_ω ; then

$$H_p(X, \mathcal{G}_\omega^\vee) = 0 \quad \text{for } p \neq n,$$

$$H_n(X, \mathcal{G}_\omega^\vee) = \sum C \cdot \Delta_\nu(\omega) \quad (\text{direct sum})$$

where summation runs over the twisted cycles
associated with $\binom{m-1}{n}$ bounded chambers.

4. A perfect pairing.

$$H_n(X, \mathcal{G}_\omega^\vee) \times H^n(X, \mathcal{G}_\omega) \xrightarrow{\text{a perfect pairing}} \mathbb{C}$$

||

$$\sum_\nu C \cdot \Delta_\nu(\omega) \times \{ \{ \varphi \langle j_1, \dots, j_n \rangle \mid 1 \leq j_1 < \dots < j_n \leq m-1 \} \} \longrightarrow \mathbb{C}$$

$$(\sigma, \varphi) \longrightarrow \int_\sigma U \cdot \varphi$$

That the above pairing is perfect is equivalent to

$$\det \left(\int_{\Delta_\nu(\omega)} U \cdot \varphi \langle J \rangle \right) \neq 0.$$

§3. The Wronskian of the HG function of type $(n+1, m+1)$.

3.1. To show the linear independence of the HG integrals

$$\int_{\Delta_\nu(\omega)} U \cdot \varphi \langle 1, \dots, n \rangle \quad (1 \leq \nu \leq \binom{m-1}{n}),$$

we must make a proper choice of partial derivatives of the integrals and prove that the Wronskian is not zero. To give a better understanding of the paper, we begin by illustrating our idea by some important examples.

Example 1. $E(2, 3+l)$ Appell's F_1 ($l=2$) Lauricella's F_D ($l \geq 3$)

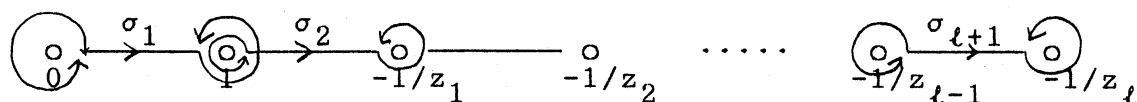
$$W = \begin{pmatrix} 1 & 0 & 1 & \dots & 1 & 1 \\ 0 & 1 & z_1 & \dots & z_l & -1 \end{pmatrix} \in M(2, 3+l; \mathbb{R})$$

$$z = (z_1, \dots, z_\ell)$$

Set

$$f_1 = u, \quad f_2 = 1 + z_1 u, \quad \dots, \quad f_{\ell+1} = 1 + z_\ell u, \quad f_{\ell+2} = 1 - u$$

$$U = \prod_{j=1}^{\ell+2} f_j^{\lambda_j}, \quad \omega = d \log U, \quad \varphi\langle i \rangle = df_i / f_i$$



$$\sigma_1, \dots, \sigma_{\ell+1} : \text{a basis of } H_1(X, g_\omega^\vee)$$

Let $\sigma \in H_1(X, g_\omega^\vee)$ and let

$$\begin{aligned} F(z) &= \int_{\sigma} U \cdot \varphi\langle 1 \rangle \\ &= \int_{\sigma} u^{\lambda_1} (1 + z_1 u)^{\lambda_2} \dots (1 + z_\ell u)^{\lambda_{\ell+1}} (1 - u)^{\lambda_{\ell+2}} \frac{du}{u} \end{aligned}$$

be a HG integral; then we have

$$\frac{\partial F}{\partial z_1} = \lambda_2 \int_{\sigma} U \cdot \frac{du}{1 + z_1 u} = \frac{\lambda_2}{\lambda_1} \int_{\sigma} U \cdot \varphi\langle 2 \rangle,$$

$$\frac{\partial F}{\partial z_2} = \frac{\lambda_3}{\lambda_2} \int_{\sigma} U \cdot \varphi\langle 3 \rangle, \quad \dots, \quad \frac{\partial F}{\partial z_\ell} = \frac{\lambda_{\ell+1}}{\lambda_\ell} \int_{\sigma} U \cdot \varphi\langle \ell+1 \rangle.$$

Let

$$W = \begin{vmatrix} \int_{\sigma_1} U \cdot \varphi\langle 1 \rangle & \dots & \int_{\sigma_{\ell+1}} U \cdot \varphi\langle 1 \rangle \\ \frac{\partial}{\partial z_1} \int_{\sigma_1} U \cdot \varphi\langle 1 \rangle & \dots & \frac{\partial}{\partial z_1} \int_{\sigma_{\ell+1}} U \cdot \varphi\langle 1 \rangle \\ \vdots & & \vdots \end{vmatrix}$$

$$\left| \begin{array}{cccc} \frac{\partial}{\partial z_l} \int_{\sigma_1} U \cdot \varphi \langle 1 \rangle & \dots & \frac{\partial}{\partial z_l} \int_{\sigma_{l+1}} U \cdot \varphi \langle 1 \rangle & \end{array} \right|$$

be the Wronskian of $l+1$ HG integrals $\int_{\sigma} U \cdot \varphi \langle 1 \rangle$; then we have

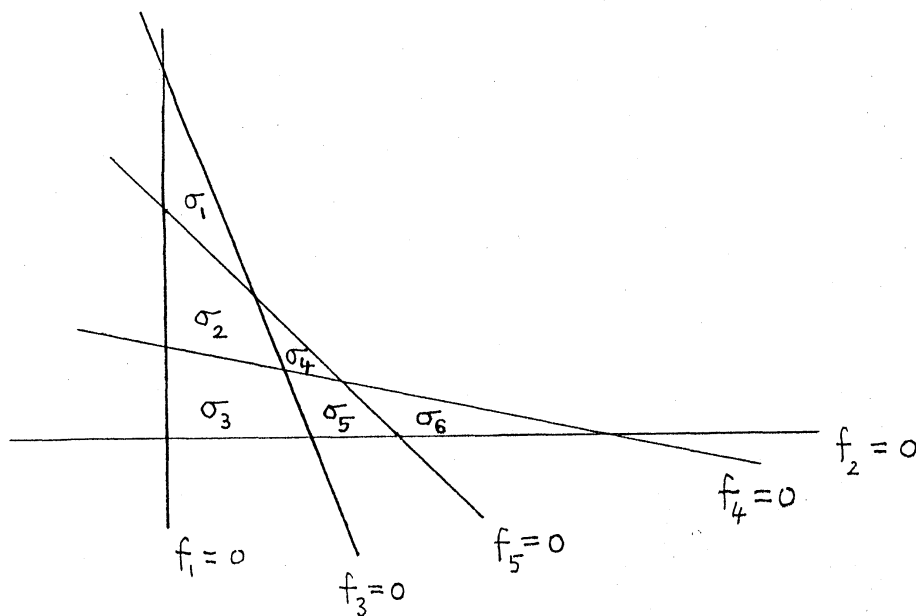
$$W = \frac{\lambda_2 \cdots \lambda_{l+1}}{z_1 \cdots z_l} \det \left(\int_{\sigma_v} U \cdot \varphi \langle j \rangle \right) \neq 0$$

because of the perfect pairing.

Example 2. $E(3,6)$.

$$w = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & z_{11} & z_{12} & -1 \\ & & 1 & z_{21} & z_{22} & -1 \end{pmatrix} \in M(3,6;R),$$

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.$$



Set

$$f_1 = u_1, \quad f_2 = u_2, \quad f_3 = 1 + z_{11}u_1 + z_{21}u_2, \quad f_4 = 1 + z_{12}u_1 + z_{22}u_2,$$

$$f_5 = 1 - u_1 - u_2,$$

$$U = \prod_{j=1}^5 f_j^{\lambda_j}, \quad \omega = d \log U$$

$$\varphi_{\langle ij \rangle} = \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}.$$

Let $\sigma \in H_2(X, \mathcal{G}_\omega^\vee)$ and let

$$\begin{aligned} F(z) &= \int_{\sigma} U \cdot \varphi_{\langle 12 \rangle} \\ &= \int_{\sigma} u_1^{\lambda_1} u_2^{\lambda_2} (1+z_{11}u_1+z_{21}u_2)^{\lambda_3} (1+z_{12}u_1+z_{22}u_2)^{\lambda_4} \\ &\quad \cdot (1-u_1-u_2)^{\lambda_5} \frac{du_1}{u_1} \wedge \frac{du_2}{u_2} \end{aligned}$$

be a HG integral. We have

$$\begin{aligned} \frac{\partial F}{\partial z_{11}} &= \int_{\sigma} \lambda_3 u_1 \frac{U}{1+z_{11}u_1+z_{21}u_2} \frac{du_1}{u_1} \wedge \frac{du_2}{u_2} \\ &= \frac{\lambda_3}{z_{11}} \int_{\sigma} U \cdot \frac{d(1+z_{11}u_1+z_{21}u_2)}{1+z_{11}u_1+z_{21}u_2} \wedge \frac{du_2}{u_2} = -\frac{\lambda_3}{z_{11}} \int_{\sigma} U \cdot \varphi_{\langle 23 \rangle}, \end{aligned}$$

$$\frac{\partial F}{\partial z_{12}} = -\frac{\lambda_4}{z_{12}} \int_{\sigma} U \cdot \varphi_{\langle 24 \rangle}, \quad \frac{\partial F}{\partial z_{21}} = \frac{\lambda_3}{z_{21}} \int_{\sigma} U \cdot \varphi_{\langle 13 \rangle},$$

$$\frac{\partial F}{\partial z_{22}} = \frac{\lambda_4}{z_{22}} \int_{\sigma} U \cdot \varphi_{\langle 14 \rangle}, \quad \frac{\partial^2 F}{\partial z_{11} \partial z_{22}} = \frac{\lambda_3 \lambda_4}{\det z} \int_{\sigma} U \cdot \varphi_{\langle 34 \rangle}.$$

Let

$$W = \left| \begin{array}{ccc} \int_{\sigma_1} U \cdot \varphi_{\langle 12 \rangle} & \dots & \int_{\sigma_6} U \cdot \varphi_{\langle 12 \rangle} \\ \frac{\partial}{\partial z_{11}} \int_{\sigma_1} U \cdot \varphi_{\langle 12 \rangle} & \dots & \frac{\partial}{\partial z_{11}} \int_{\sigma_6} U \cdot \varphi_{\langle 12 \rangle} \end{array} \right|$$

$$\begin{vmatrix} \vdots & & \vdots \\ \frac{\partial}{\partial z_{22}} \int_{\sigma_1} U \cdot \varphi \langle 12 \rangle & \dots & \frac{\partial}{\partial z_{22}} \int_{\sigma_6} U \cdot \varphi \langle 12 \rangle \\ \frac{\partial^2}{\partial z_{11} \partial z_{22}} \int_{\sigma_1} U \cdot \varphi \langle 12 \rangle & \dots & \frac{\partial^2}{\partial z_{11} \partial z_{22}} \int_{\sigma_6} U \cdot \varphi \langle 12 \rangle \end{vmatrix}$$

be the Wronskian of 6 HG integrals $\int_{\sigma_v} U \cdot \varphi \langle 12 \rangle$; then we have

$$W = \left(-\frac{\lambda_3}{z_{11}} \right) \left(-\frac{\lambda_4}{z_{12}} \right) \left(\frac{\lambda_3}{z_{21}} \right) \left(\frac{\lambda_4}{z_{22}} \right) \left(\frac{\lambda_3 \lambda_4}{\det z} \right) \times \det \left(\int_{\sigma_v} U \cdot \varphi \langle ij \rangle \right).$$

On the other hand, $\{ \varphi \langle ij \rangle \mid 1 \leq i < j \leq 4 \}$ forms a basis of $H^2(X, \nabla_\omega)$ and hence the perfect pairing yields

$$W = \frac{(\lambda_3 \lambda_4)^3}{2 \prod_{i,j=1}^4 z_{ij} \cdot \det z} \det \left(\int_{\sigma_v} U \cdot \varphi \langle ij \rangle \right) \neq 0.$$

3.2. General case.

$$w = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ & 1 & z_{1,n+1} & \dots & z_{1,m-1} & -1 \\ & & \vdots & & \vdots & \vdots \\ & & 1 & z_{n,n+1} & \dots & z_{n,m-1} & -1 \end{pmatrix} \in M(n+1, m+1; R).$$

We set

$$z = \begin{pmatrix} z_{1,n+1} & \dots & z_{1,m-1} \\ \vdots & & \vdots \\ z_{n,n+1} & \dots & z_{n,m-1} \end{pmatrix} \in M(n, m-n-1; R)$$

& for simplicity,

$$z \begin{pmatrix} i_1 i_2 & \cdots & i_p \\ k_1 k_2 & \cdots & k_p \end{pmatrix} = \det \begin{pmatrix} z_{i_1 k_1} & \cdots & z_{i_1 k_p} \\ \vdots & & \vdots \\ z_{i_p k_1} & \cdots & z_{i_p k_p} \end{pmatrix},$$

$$\varphi \langle j_1, \dots, j_n \rangle = \frac{df_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{df_{j_n}}{f_{j_n}},$$

$$F(\lambda, z) = \int_{\sigma} U \cdot \varphi \langle 12 \cdots n \rangle \quad \text{where } \sigma \in H_n(X; g_{\omega}^V).$$

Set $I = \{i_1, \dots, i_p\}$, $K = \{k_1, \dots, k_p\}$, $H = \{n+1, \dots, m\} \setminus K$

$$L = \{1, \dots, n\} \setminus I \underset{\text{set}}{=} \{\ell_1, \dots, \ell_{n-p}\}$$

where we suppose

$$1 \leq i_1 < \cdots < i_p \leq n, \quad n+1 \leq k_1 < \cdots < k_p \leq m-1, \quad 1 \leq \ell_1 < \cdots < \ell_{n-p} \leq n.$$

Then we get

$$U = \prod_{i=1}^n u_i^{\lambda_i} \cdot \prod_{j=n+1}^{m-1} (u_1 z_{1j} + \cdots + u_n z_{nj})^{\lambda_j} \cdot (1 - \sum u_i)^{\lambda_m}$$

$$\begin{aligned} \frac{\partial^p F}{\partial z_{i_1 k_1} \cdots \partial z_{i_p k_p}} &= \lambda_{k_1} \cdots \lambda_{k_p} \int_{\sigma} u_{i_1} \cdots u_{i_p} \cdot \frac{1}{\prod_{k \in K} f_k} U \cdot \varphi \langle 1 \cdots n \rangle \\ &= \prod_{k \in K} \lambda_k \int_{\sigma} U \cdot \frac{1}{u_{\ell_1} \cdots u_{\ell_{n-p}} \prod_{k \in K} f_k} du_1 \wedge \cdots \wedge du_n. \end{aligned}$$

Since

$$\begin{aligned} &du_{\ell_1} \wedge \cdots \wedge du_{\ell_{n-p}} \wedge df_{k_1} \wedge \cdots \wedge df_{k_p} \\ &= du_{\ell_1} \wedge \cdots \wedge du_{\ell_{n-p}} \wedge z \begin{pmatrix} i_1 & \cdots & i_p \\ k_1 & \cdots & k_p \end{pmatrix} du_{i_1} \wedge \cdots \wedge du_{i_p} \end{aligned}$$

$$= z \binom{I}{K} \operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & n \\ L & & & I \end{pmatrix} du_1 \wedge \cdots \wedge du_n$$

$$\dots \frac{\partial^p F}{\partial z_{i_1 k_1} \cdots \partial z_{i_p k_p}} = \frac{\operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & n \\ L & & & I \end{pmatrix}}{z \binom{I}{K}} \left(\prod_k \lambda_k \right) \int_{\sigma} U \cdot \varphi \langle LK \rangle.$$

On the other hand, since

$$1 \leq i_1 < \cdots < i_{n-p} \leq n, \quad n+1 \leq k_1 < \cdots < k_p \leq m-1,$$

$\{\varphi \langle LK \rangle\}$ is a subset of the basis $\{\varphi \langle j_1 \cdots j_n \rangle \mid 1 \leq j_1 < \cdots < j_n \leq m-1\}$.

Since

$$\#\{\varphi \langle LK \rangle\} = \sum_{p=0}^n \binom{n}{n-p} \binom{m-n-1}{p} = \binom{m-1}{n} = \dim H^n(X, \nabla_{\omega}),$$

we see that $\{\varphi \langle LK \rangle\}$ coincides with the basis $\{\varphi \langle j_1 \cdots j_n \rangle\}$. Put $N = \binom{m-1}{n}$

and let $\sigma_1, \dots, \sigma_N$ be the twisted n -cycles associated with the N bounded chambers, which form a basis of $H_n(X, \mathcal{G}_{\omega}^{\vee})$. Set

$$F_{\nu}(\lambda, x) = \int_{\sigma_{\nu}} U \cdot \varphi \langle 1 \cdots n \rangle;$$

then

$$\begin{aligned} W &= \det \left(\frac{\partial^p F_{\nu}}{\partial z_{i_1 k_1} \cdots \partial z_{i_p k_p}} \right)_{\substack{\nu=1, \dots, N \\ I, K}} \\ &= \left(\prod_{I, K} \frac{\operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & n \\ L & & & I \end{pmatrix}}{z \binom{I}{K}} \lambda_{k_1} \cdots \lambda_{k_p} \right) \times \det \left(\int_{\sigma_{\nu}} U \cdot \varphi \langle LK \rangle \right). \end{aligned}$$

By the perfect pairing of $H^n(X, \nabla_{\omega})$ and $H_n(X, \mathcal{G}_{\omega}^{\vee})$, we have showed that

$\det \left(\int_{\sigma_{\nu}} U \cdot \varphi \langle LK \rangle \right) \neq 0$ and hence the Wronskian $W \neq 0$ if each $z \binom{I}{K} \neq 0$.

Remark Varchenko[V1,2] showed that $\det(\int_{\sigma_v} U \cdot \phi \langle LK \rangle)$ can be written in closed form as a product of a generalized beta function and critical values of $f_j^{\lambda_j}$ on bounded chambers Δ .

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